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# An algorithm for the active stabilization of a spacecraft with viscoelastic elements under conditions of uncertainty☆

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**Abstract**

An algorithm for stabilizing the angular position of a spacecraft with dynamic elastic elements possessing dissipative properties is proposed, based on bang-bang feedback with delay. The control is achieved under conditions of uncertainty. The accuracy of the equation is demonstrated by a numerical example.


The problem of controlling a rigid body with flexible elements was actively investigated in the last quarter of the last century. The problem of synthesizing a control which ensures asymptotic stability of the solutions of the system and also optimality of the controls has been considered within the framework of linear theory.[1–3] The effect of viscous damping on the stability of the control process was investigated in Ref. 4. The problem of stabilizing artificial satellites with viscoelastic panels of solar batteries was solved in problems of the mechanics of space flight in Refs. 5,6.

Recently a range of problems of the mechanics of space flight, solved over a long time interval, taking into account parametric uncertainty, unknown external perturbations, uncertain delay time in the operating and observing systems, incomplete observability of the system, the discreteness of modern digital control systems, the need to damp oscillations of elastic components, fast and slow displacements for systems with an infinite number of degrees of freedom and the effect of gravitational forces, has recently been investigated and considerably extended.

A new successful scheme for controlling spacecraft under conditions of parametric uncertainty, weak natural damping of the elastic oscillations of the structure, discrete measurements of only accessible coordinates and the presence of delay when setting up the equation was recently proposed in Ref. 7. A bang-bang profile control algorithm was developed in Ref. 8, which reduces the amplitude of the oscillations of elastic elements in the orientation of the spacecraft. The evolution of the motion of an artificial satellite with viscoelastic rods in a circular orbit, taking the effect of gravitational forces into account, was investigated in Ref. 9.

Below we consider the problem of stabilizing the angular position of a spacecraft with two dynamically elastic components about its centre of mass using jet engines possessing unknown delay. Two new bang-bang control algorithms are proposed, taking into account the uncertain delay and external perturbations. One algorithm is obtained by synthesizing a traditional two-pulse optimal control and a local bang-bang stabilization algorithm.[10] The other algorithm provides non-local stabilization of the system.[11] Similar problems were considered previously in Refs. 10–14.

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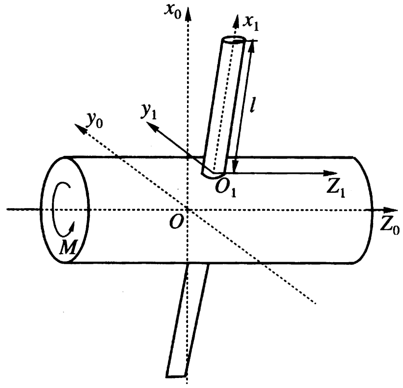


Fig. 1.

## 1. Formulation of the problem

Consider a spacecraft, represented by a rigid body with two dynamic elastic elements (rods) (Fig. 1). We will assume that identical viscoelastic rods are fastened symmetrically and that the oscillations of the elastic structure are small. We will only consider the controlled motion around the longitudinal axis of the spacecraft. Suppose the rods execute antisymmetric oscillations. We can therefore confine ourselves to considering only one rod, and the effect of their interaction on the main body is doubled. A similar problem has already been considered in the literature (see, for example, Ref. 2).

We will introduce the following notation: $r$ is the distance from the longitudinal axis to the point where the rod is fastened, $l$ is the rod length, EI is the flexural stiffness of the rod, $\chi$ is the coefficient of internal viscous friction, $m$ is the mass per unit length of the rod, $J_0$ is the moment of inertia of the spacecraft about the $OZ$ axis and $M$ is the control moment applied to the spacecraft.

Suppose the first three systems of coordinates are defined as follows: $Ox_0y_0z_0$ is the inertial system of coordinates with origin $O$ at the centre of mass of the mechanical system, $Ox_1y_1z_1$ is a system of coordinates rigidly connected to the spacecraft with origin at the point $O$, and $O_1xyz$ is a system of coordinates connected with the undeformed rod with origin at the point $O_1$.

The position of the system $Ox_1y_1z_1$ is defined by the angle of rotation $\gamma(t)$ of the spacecraft. The deflection of the rod from the $O_1x$ axis will be denoted by $y(x, t)$ (Fig. 2).

Considering the action integral and taking its variation into account, we obtain by standard methods the following equations of the free oscillations of the system

$$J\ddot{\gamma}(t) + 2\int_0^l m(x+r)\frac{\partial^2 y(x,t)}{\partial t^2}dx = 0, \quad J = J_0 + 2\int_0^l m(x+r)^2 dx$$

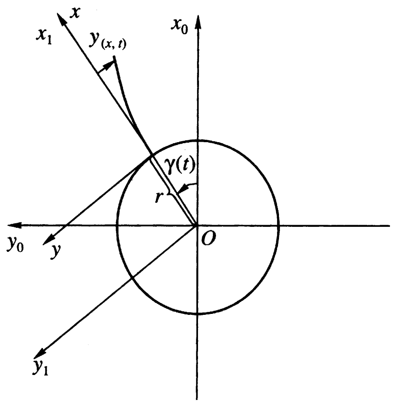


Fig. 2.

$$m(x+r)\ddot{\gamma}+m\frac{\partial^2 y(x,t)}{\partial t^2}+\mathrm{EI}\frac{\partial^4 y(x,t)}{\partial x^4} = 0$$

with boundary conditions

$$y(0,t) = \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=l} = \left.\frac{\partial^3 y(x,t)}{\partial x^3}\right|_{x=l} = 0.$$

Here $J$ is the moment of inertia of the whole system.

If control is achieved by means of a torque $u(t-h(t))$, applied to the body, the rods possess dissipative properties and external perturbations $f$ act on the rigid body, the equations of free oscillations of the system take the form

$$J\ddot{\gamma}(t)+2\int_0^l m(x+r)\frac{\partial^2 y(x,t)}{\partial t^2}dx = u(t-h(t))+f \tag{1.1}$$

$$m(x+r)\ddot{\gamma}+m\frac{\partial^2 y(x,t)}{\partial t^2}+\mathrm{EI}\frac{\partial^4 y(x,t)}{\partial x^4}+\mathrm{EI}\chi\frac{\partial^5 y(x,t)}{\partial t\partial x^4} = 0. \tag{1.2}$$

Using Bubnov's method, we can assume approximately that

$$y(x,t) = \sum_{i=1}^k q_i(t)\Phi_i(x), \tag{1.3}$$

where $\Phi_i(x)$ is the natural form corresponding to the positive eigenvalue $\lambda_i$ of the positive self-conjugate operator

$$L\Phi(x) = \frac{d^4\Phi(x)}{dx^4},\quad \Phi(0) = \Phi'(0) = \Phi''(l) = \Phi'''(l) = 0,$$

where

$$\frac{d^4\Phi_i(x)}{dx^4}\equiv\lambda_i\Phi_i(x),\quad 0\le x\le l.$$

We substitute representation (1.3) into Eqs. (1.1) and (1.2). We then multiply the equation obtained on substitution into Eq. (1.2) by $\Phi_j(x)$ and integrate over the section $[0, l]$, using the property of orthogonality of the eigenfunctions. As a result we obtain a system of ordinary differential equations for determining $\gamma$ and $q_i$

$$J\ddot{\gamma}+2\sum_{i=1}^k p_i\ddot{q}_i = u(t-h(t))+f \tag{1.4}$$

$$p_i\ddot{\gamma}+a_i\ddot{q}_i+b_i\dot{q}_i+c_iq_i = 0,\quad i = 1,2,\dots,k, \tag{1.5}$$

where

$$p_i = \int_0^l m(x+r)\Phi_i(x)dx,\quad a_i = m\kappa_i,\quad b_i = \lambda_i\mathrm{EI}\chi\kappa_i,\quad c_i = \lambda_i\mathrm{EI}\kappa_i;\quad \kappa_i = \int_0^l\Phi_i^2(x)dx,$$

and the quantity $f$ now represents not only the external perturbations but also the inaccuracy of the model, connected with the approximate representation of $y(x, t)$.

We will put

$$\beta = J\gamma+2\sum_{i=1}^k p_iq_i.$$

In this case system (1.4), (1.5) is equivalent to the system

$$\ddot{\beta} = u(t-h(t)) + f, \quad (JA - 2\mathbf{pp}^T)\ddot{\mathbf{q}} + JB\dot{\mathbf{q}} + JC\mathbf{q} = -\mathbf{p}(u(t-h(t)) + f), \tag{1.6}$$

where

$$\mathbf{q} = (q_1, \ldots, q_k)^T, \quad \mathbf{p} = (p_1, \ldots p_k)^T$$

$$A = \mathrm{diag}\{a_1, \ldots, a_k\}, \quad B = \mathrm{diag}\{b_1, \ldots, b_k\}, \quad C = \mathrm{diag}\{c_1, \ldots, c_k\}.$$

It is important to note that the variables β and **q** are separable here.

Suppose the symmetric matrix JA - 2$\mathbf{pp}^T$ is positive definite. It can be shown that the unperturbed open system for system (1.6), ignoring dissipation, has a spectrum consisting of a single zero root of multiplicity 2 and $k$ complex conjugate pairs, lying on the imaginary axis. (For this it is sufficient to use certain assertions from linear algebra on regular beams of quadratic forms[15].) In the case of slight dissipation in the elastic elements the complex-conjugate pairs are shifted into the left complex half-plane, thereby ensuring the stability of the subsystem corresponding to the elastic elements, while the zero multiple root, as before, remains unchanged.

We will construct an equation which ensures that the quantity β is stabilized. By virtue of the stability of the subsystem for the elastic elements for small perturbations and the small control for long times, the oscillations of the elastic elements will be small.

It should be noted that the quantities $q_i$ are often not directly observable, and hence to synthesize the control with feedback with respect to the variable β it may be necessary to construct an observer.

## 2. Attractive control

We will put $\nu = \dot{\beta}$ and in the phase plane (β, $\nu$) we will investigate the dynamic flux

$$\dot{\beta} = \nu, \quad \dot{\nu} = u, \tag{2.1}$$

where $u$ can take only two values, namely, +1 and −1. Suppose that, in a certain time interval $[t_0, t_1]$, the identity $u \equiv 1$ holds. Then, system (2.1) has the form

$$d\beta/dt = \nu, \quad d\nu/dt = 1.$$

Hence it follows that $\nu^2 - 2\beta = c_1$. When $u = -1$ in the section $[t_0, t_1]$ we have

$$d\beta/dt = \nu, \quad d\nu/dt = -1, \quad \nu^2 + 2\beta = c_2.$$

Hence, in the phase plane (β, $\nu$), motion occurs along two families of parabolae

$$\nu^2 - 2\beta = c_1, \quad u = 1; \quad \nu^2 + 2\beta = c_2, \quad u = -1.$$

We will distinguish two branches of the parabolae

$$\Gamma^+ : \nu^2 + 2\beta = 0, \quad \nu \geq 0; \quad \Gamma^- : \nu^2 - 2\beta = 0, \quad \nu \leq 0.$$

Finally, in the phase plane (β, $\nu$) we define two regions of initial data

$$G^+ = \begin{cases} \nu^2 + 2\beta \geq 0, & \nu \geq 0 \\ \nu^2 - 2\beta < 0, & \beta > 0, \end{cases} \quad G^- = \begin{cases} \nu^2 + 2\beta \leq 0, & \beta \leq 0 \\ \nu^2 - 2\beta > 0, & \nu < 0. \end{cases}$$

The attractive control $u$ will only be determined by the fact that the initial point $(\beta_0, \nu_0)$ belongs either to the region $G^+$ or $G^-$. Suppose $(\beta_0, \nu_0) \in G^+$ (Fig. 3). Then the point $(\beta(t), \nu(t))$ in the initial time interval $(0, T_1)$ moves along the parabola

$$\nu^2 + 2\beta = \gamma_0; \quad \gamma_0 = \nu_0^2 + 2\beta_0$$

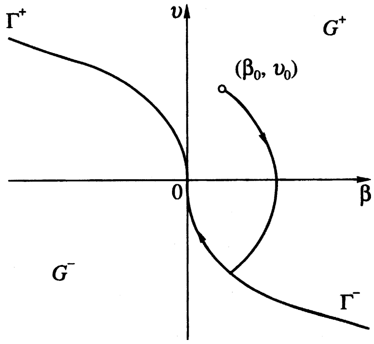


Fig. 3.

up to the instant $T_1$ where it meets the curve $\Gamma$ at the point $(\beta_1, \nu_1)$. Simple calculation shows that

$$\beta_1 = \gamma_0/4, \quad \nu_1 = -\sqrt{\gamma_0/2}.$$

In this case $T_1 = \nu_0 + \sqrt{\gamma_0/2}$. In the second interval $(T_1, T_2)$ the phase trajectory moves along the curve $\Gamma^-$ to the left and arrives at the origin of coordinates after a time $T_2 - T_1$; here $T_2 = \nu_0 + 2\sqrt{\gamma_0/2}$. Henceforth we must put $u = 0$. Hence, for $(\beta_0, \nu_0) \in G^+$

$$u = \begin{cases} 1, & 0 \le t \le T_1 \\ -1, & T_1 < t \le T_2 \\ 0, & t > T_2. \end{cases} \tag{2.2}$$

The qualitative picture is similar for the case $(\beta_0, \nu_0) \in G^-$.

## 3. Local stabilization

Since the control $u$ has a certain delay and unknown external perturbations act on the system, for small $h(t)$ and $f(t)$, using Eq. (2.2), one need not take the system as being strictly at the origin of coordinates but in a certain small neighbourhood of it. Hence, for the case of a retaining control one can assume that the initial point $(\beta_0, \nu_0)$ is in the vicinity of the origin of coordinates, and we can consider the system

$$\dot{\beta} = \nu, \quad \dot{\nu} = u(t - h(t)) + f(t), \quad t > t_0 = T_2. \tag{3.1}$$

The control $u$ will be sought in the form

$$u(t - h(t)) = -p\,\mathrm{sign}[c\beta(t - h(t)) + \nu(t - h(t))].$$

The parameters $p$, $c > 0$ will be indicated below.

Making the change of variables $s = c\beta + \nu$ in system (3.1) (see Ref. 16), we will have

$$\dot{\beta} = -c\beta + s, \quad \dot{s} = cs - p\,\mathrm{sign}[s(t - h(t))] - c^2\beta + f(t). \tag{3.2}$$

Suppose $\varepsilon > 0$ is an arbitrary number, defining the radius of the neighbourhood of the origin of coordinates within which the solution must be contained. We will assume that $0 < h(t) < h_0$ and the parameters $c$ and $p$ are chosen as follows:

$$0 < c < \frac{1}{h_0}\ln\frac{6}{5}, \quad p = \frac{3}{2}c^2\varepsilon.$$

**Theorem 1.** *If*

$$|\beta_0| < \varepsilon/4, \quad |c\beta_0 + \nu_0| < \delta_0 = c\varepsilon(6 - 5e^{ch_0})/(4e^{ch_0}), \quad |f(t)| < c^2\varepsilon/4,$$

*then* $|\beta(t)| < \varepsilon$ *for all* $t > t_0$.

**Proof.** Suppose $\varepsilon_0 = c\varepsilon/2$, then

$$\delta_0 = \varepsilon_0(6-5e^{ch_0})/(2e^{ch_0}).$$

We substitute the quantity

$$\beta(t) = \beta_0 e^{-c(t-t_0)} + \int_{t_0}^{t} e^{-c(t-\tau)} s(\tau)d\tau \tag{3.3}$$

into the second equation of (3.2). We will first show that $|s(t)| < \varepsilon_0$ for any $t > t_0$. Let us assume the opposite. Then an instant of time $T > t_0$ exists, such that $|s(T)| = \varepsilon_0$. Obviously

$$|s(t_0)| = |c\beta_0 + \nu_0| < \delta_0 < \varepsilon_0$$

and hence the last instant of time $t^* \in [t_0, T)$ exists at which $s(t^*) = \delta_0$ and $\delta_0 < s(t) < \varepsilon_0$ for all $t \in [t^*, T]$. We will prove that in this case $T - t^* > h_0$. For this we will first estimate the right-hand side of the second Eq. of (3.2) in the interval $[t_0, T]$. We have

$$\dot{s} \le cs + p + |f(t)| + c^2\left(|\beta_0| + \int_{t_0}^{t} e^{-c(t-\tau)}|s(\tau)|d\tau\right) \le cs + 5c\varepsilon_0.$$

Hence

$$s(t) \le (\delta_0 + 5\varepsilon_0)e^{c(t-t^*)} - 5\varepsilon_0, \quad t \ge t_0.$$

Therefore for $t = T$, taking into account the fact that

$$\delta_0 < \varepsilon_0(6-5e^{ch_0})/e^{ch_0}$$

we obtain

$$T - t^* > \frac{1}{c}\ln\frac{6\varepsilon_0}{\delta_0 + 5\varepsilon_0} > h_0.$$

Then, for $t \in [t^* + h_0, T)$ we have $\operatorname{sign}[s(t - h(t))] = 1$, and so

$$\dot{s} \le cs - p + 2c\varepsilon_0 \le c\varepsilon_0 - p + 2c\varepsilon_0 = 0.$$

Therefore, for all $t > t_0$ the inequality $|s(t)| < \varepsilon_0$ holds. It only remains to note that $|\beta(t)| < |\beta_0| + \varepsilon_0/c < \varepsilon$. □

It should be noted that Theorem 1 does not give representations on the qualitative behaviour of the solution of the system in the ε-neighbourhood of zero. Whether the solution will tend asymptotically to a certain equilibrium position or whether a limit cycle or a strange attractor will arise is unknown. However, numerical results show that such systems often have oscillatory solutions, and the amplitude of the oscillations depends considerably on the coefficient $p$ preceding the bang-bang element.

## 4. Non-local stabilization

The idea of the non-local stabilization algorithm. For the equation

$$\dot{x}(t) = \lambda x(t) - p_0 \operatorname{sign}[x(t-h_0)]; \quad x(t) = \varphi(t), \quad t \in [t_0 - h_0, t_0], \tag{4.1}$$

where $x \in R$ is a variable state, $\lambda$, $p_0 > 0$ are the parameters of the system and $h_0 > 0$ is the delay time, certain important properties of its solution were established in Ref. 17: system (4.1) is stabilizable if $\lambda h_0 < \ln 2$, all the non-zero stable solutions of system (4.1) are oscillatory, and the radius of the neighbourhood of stable oscillations is proportional to the gain $p_0$.

These properties lead to the conclusion that we can attempt to suggest the algorithm for adapting $p_0$ preceding the bang-bang element, which will enable us to extend the range of initial data and increase the accuracy of the control.

It was suggested in Refs. 12 and 13 that the following should be chosen

$$p_0 = -p\left(1+2\sum_{n=1}^{N}3^{n-1}H_n(|x(t-h(t))|)\right);\quad H_n(\varrho) = \begin{cases}1, & \text{if } \varrho \ge \nu_n \\ 0, & \text{if } \varrho < \nu_n,\end{cases}\quad \nu_n > 0$$

where $p$ and $N$ are control parameters and $H_n(\varrho)$ is the Heaviside function.

As it applies to the problem of controlling a spacecraft, this control can be interpreted as the possibility of operating with several modes of operation of the jet engine or as the presence in the spacecraft of several engines of different power.

The synthesis of bang-bang control for the non-local case. We will consider (3.1) and attempt to modernize the control $u(t-h(t))$, starting from the discussions presented above. We will choose arbitrary $R$ and $\varepsilon(R>\varepsilon>0)$. We will assume that $|\beta_0|<R$ and $|\nu_0|<R$. It is required to construct a control which stabilizes the solution of system (3.1) in the $\varepsilon$-neighbourhood of zero, $|\beta(t)|<\varepsilon$, $|\nu(t)|<\varepsilon$ for all $t>T$, where $T$ is a certain positive number.

We will construct the control in accordance with the following algorithm.

1°. We determine the parameters $c$, $\gamma$ and $p$ such that the following relations are satisfied

$$0 < ch_0 < \ln\frac{15}{14},\quad \frac{e^{ch_0}}{10-9e^{ch_0}} < \gamma < 3,\quad p = \frac{15}{4}c^2\varepsilon\gamma.$$

2°. We fix

$$\varepsilon_0 = 3c\varepsilon/4,\quad R_0 = (c+1)R.$$

3°. We choose an integer $N$ such that it satisfies the inequality

$$\log_3(R_0/\varepsilon_0) \le N < \log_3(R_0/\varepsilon_0)+1.$$

4°. We determine the sequence $\nu_1<\nu_2<\ldots<\nu_{N+1}$ from the formula

$$\nu_n = 3^{n-1}\gamma\varepsilon_0.$$

5°. Finally, we choose the control in the form

$$\begin{gathered}u(t-h(t)) = -A_N\operatorname{sign}[c\beta(t-h(t))+\nu(t-h(t))] \\ A_N = p\left[1+2\sum_{n=1}^{N}3^{n-1}H_n(|c\beta(t-h(t))+\nu(t-h(t))|)\right].\end{gathered} \tag{4.2}$$

**Theorem 2.** *For any initial data $|\beta_0|<R$ and $|\nu_0|<R$ and any limited perturbation $|f(t)|<\nu_1c/2$, a control $u$ of the form (4.2) will guarantee the existence of an instant of time $T_\varepsilon>t_0$, beginning from which the following inequalities will be satisfied*

$$|\beta(t)|<\varepsilon,\quad |\nu(t)|<\varepsilon.$$

**Proof.** We will use several properties of the solution of the system

$$\dot\beta = -c\beta+s,\quad \dot s = cs+u(t-h(t))-c^2\beta+f(t) \tag{4.3}$$

with a control of the form (4.2), the proofs of which are presented in the Appendix.□

**Property 1.** *We will introduce the function*

$$I(t) = Re^{-c(t-t_0)} + \int_{t_0}^{t} e^{-c(t-\tau)}|s(\tau)|d\tau.$$

*If* $|s(t)| < \rho$ *for all t greater than a certain* $t_\rho$*, an instant of time* $\bar{t}_\rho > t_\rho$ *exists such that* $I(t) < (7\rho)/(6c)$ *when* $t > \bar{t}_\rho$.

**Property 2.** *If, at a certain instant of time* $T_k > t_0$ *for the solution of system* (4.3) *the inequalities* $|s(T_k)| < \nu_k/\gamma$, $|s(t)| < \nu_k$, *are satisfied for* $t \in [T_k - h_0, T_k]$ *and* $I(t) < (7\nu_{k+1})/(6c)$ for $t > T_k$, then $|s(t)| < \nu_k$ for any $t > t_k$.

**Property 3.** *For any finite* $T > t_0$ *there is a* $T_0 > T$ *such that* $s(T_0) = 0$.

**Property 4.** *If* $|s(t)| < \nu_k$ *for all* $t > T_k$*, an instant of time* $T_{k-1} \geq T_k$ *exists such that* $|s(t)| < \nu_{k-1}$ *for all* $t > T_{k-1}$*, where* $k = 2, 3, \ldots, N+1$.

Since $|\nu_0| < R$, we have $|s(0)| = |c\beta_0 + \nu_0| < R_0$. Hence, by virtue of Property 3 we obtain $|s(t)| < \nu_{N+1}$ for all $t > 0$. According to Property 4 an instant of time $t = T_N$ exists such that the curve $s(t)$ remains in the $\nu_N$-neighbourhood of zero for all $t > T_N$, etc. At the $N$-th step we obtain that an instant of time $T_1 > 0$ exists such that $|s(t)| < \nu_1$ for all $t > T_1$.

We will show that there is a $T_\varepsilon > T_1$ such that $|s(t)| < \varepsilon_0$ for all $t > T_\varepsilon$. It follows from Property 2 that $T_0 > T_1$ exist such, $s(T_0) = 0$. We will show that $|s(t)| < \varepsilon_0$ for all $t > T_0$. If this was not so, then, at a certain instant of time $T' > T_0$ the equality $|s(T')| = \varepsilon_0$ would be satisfied. Suppose $T'$ is the first such instant of time. It is easy to see that in this case $T' - T_0 > h_0$, and so $|s(t)| < \varepsilon_0$ when $t > T_0$. Since

$$\beta(t) = \beta(0)e^{-ct} + \int_0^t e^{-c(t-\tau)}s(\tau)d\tau,$$

then, according to Property 1, a $T_\varepsilon > T_0$ exists for which $|\beta(t)| < 4c\varepsilon_0/3 = \varepsilon$ when $t > T_\varepsilon$.

It should be noted that the quantity $T_\varepsilon$ depends on the parameters $R$, $\varepsilon$ and $c$.

It can be seen that the control proposed above for large $t$ will be of the order of $\varepsilon$, and consequently, for large $t$, the coordinates of $q_i(t)$ have the same order.

## 5. Example

Consider a spacecraft of considerably non-rigid construction with small dissipation in the elastic elements, taking into account one tone of the oscillations. We will assume that

$$J_0 = 30, \quad \mathrm{EI} = 3.5, \quad m = 0.53, \quad l = 7.5, \quad r = 3, \quad k = 0.001.$$

The corresponding splitting of the system has the form

$$1183.24\ddot{\gamma} + 44.67\ddot{q} = u(t-h(t)) + f(t), \quad 22.33\ddot{\gamma} + 11.06\ddot{q} + 0.72\dot{q} + 38.83q = 0. \tag{5.1}$$

We will assume that the control possesses delay $h(t) = 0.2$ and that small external perturbations $f(t) = 0.001\sin(t)$ act on the system. It can be shown, that for the initial conditions

$$\gamma(0) = 0.7, \quad \dot{\gamma}(0) = -0.01, \quad q(0) = 0.1, \quad \dot{q}(0) = 0$$

the attractive control will have the following form

$$u = \begin{cases} 1, & 0 \leq t \leq 13.3 \\ -1, & 13.3 < t \leq 30.9. \end{cases}$$

We will define the stabilizing control in the form (4.2) with the following parameters

$$p = 0.007, \quad c = 0.31, \quad \gamma = 2.76, \quad N = 4, \quad \nu_i = 0.0045 \cdot 3^{i-1}, \quad i = 1, 2, 3, 4.$$

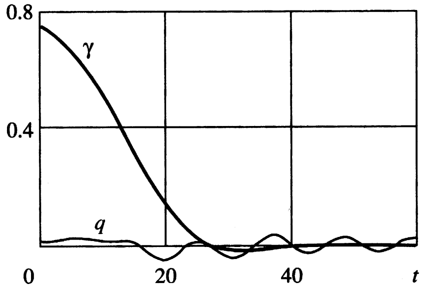


Fig. 4.

These control parameters were chosen from the following considerations: the attractive control, by virtue of the delay and the external perturbations, can operate with an error $R \le 0.1$, while the permissible stabilization error $\varepsilon = 0.007$.

In Fig. 4 we show the results of modelling for system (5.1) with the proposed control.

## Appendix A

**Proof of Properties 1–4.** Property 1 is obvious.

**Proof of Property 2.** We will assume the opposite. Then instants of time $T^* > T_k$: $s(T^*) = \nu_k$ a and $t^*$: $T_k < t^* < T^*$, exist for which

$$s(t^*) = \nu_k/\gamma; \quad \nu_k/\gamma < s(t) < \nu_k \text{ if } t \in (t^*, T^*).$$

We will first show that $T^* - t^* > h_0$. Note that

$$\dot{s} \le cs + p3^{k-1} + 7c\nu_{k+1}/6 + c\nu_1/2 \le cs + 9c\nu_k; \quad s(t^*) = \nu_k/\gamma.$$

Using Gronwall's lemma we conclude that

$$s(T^*) = \nu_k \le (\nu_k/\gamma + 9\nu_k)e^{c(T-t^*)} - 9\nu_k.$$

This leads to the inequality

$$T^* - t^* \ge \frac{1}{c}\ln\frac{10}{1/\gamma + 9}.$$

Taking into account the fact that $\gamma < 3$ and $ch_0 < \ln(15/14)$, we obtain $T^* - t^* > h_0$. Then, if $t \in [t^* + h_0, T^*]$, we will have $\operatorname{sign}[s(t - h(t))] = 1$ and

$$\dot{s} \le cs - p3^{k-1} + 7c\nu_{k+1}/6 + \nu_1 c/2 \le c\nu_k - 5c\nu_k + 4c\nu_k = 0.$$

Consequently, $s$ cannot reach the boundary $\nu_k$.

Hence, in particular, it is easy to obtain that if $\beta_0 < R$ and $s(0) < R_0$, then $|s(t)| < \nu_{N+1}$ when $t > t_0$.

**Proof of Property 3.** We will assume the opposite. Then a $T^* > t_0$ exists such that $s(t) > 0$ for all $t > T^*$. It is obvious that in this case $\operatorname{sign}[s(t - h(t))] = 1$ when $t > T^* + h_0$. It can then be shown that

$$\dot{s} \le cs - \tilde{p} + c^2 e^{-ct}\left(Re^{ct_0} + \int_{t_0}^{T^*} e^{c\tau}|s(\tau)|d\tau\right) + \frac{\nu_1 c}{2}; \quad \tilde{p} = p\left(1 + 2\sum_{n=1}^{N} H_n(|s(t-h)|)\right).$$

So, for sufficiently large $t > T_0$, we will have

$$\dot{s} \le cs - \tilde{p} + 2c\nu_1$$

and consequently $s(t) > \nu_1$, since otherwise we would obtain

$$\dot{s} < c\nu_1 - 5c\nu_1 + 2c\nu_1 = -2\nu_1 c < 0$$

and there would be an instant of time $t_0 > T$ such that $s(t_0) = 0$. Proceeding in a similar way we obtain $s(t) > \nu_{N+1}$ at the $(N+1)$-th step, which is impossible by virtue of Property 2.

**Proof of Property 4.** According to Property 1 there is an instant of time $\bar{t}_k \geq T_k$ such that $I(t) < 7\nu_k/(6c)$ for all $t > \bar{t}_k$. It then follows from Property 3 that $T_{k-1} \geq \bar{t}_k + h_0$exists with the property $s(T_{k-1}) = 0$.

We will now show that $|s(t)| < \nu_{k-1}$ for all $t > T_{k-1}$.

Assume the opposite. An instant of time $\overline{T} > T_{k-1}$ then exists such that $|s(\overline{T})| = \nu_{k-1}$. We will consider two possible cases.

1°. The inequality $|s(t)| < \nu_{k-1}$ holds for all $t \in [t, T_{k-1}]$. In this case, according to Property 2 the limit $|s(t)| < \nu_{k-1}$ holds for all $t > \bar{t}$.

2°. A $t^* : \bar{t}_k \leq t^* < T_{k-1}$ exists for which $|s(t^*)| = \nu_{k-1}$. Suppose $t^*$ is the last such instant of time. In other words, $0 < |s(t)| < \nu_{k-1})$ when $t \in (t_k^*, T_{k-1})$. Since $s(T_{k-1}) = 0$ and $|s(\overline{T})| = \nu_{k-1}$, a $T^* \in [T_{k-1}, \overline{T}]$ exists for which $|s(T^*)| = \nu_{k-1}/\gamma$. We will prove that $T^* - t^* > h_0$. In fact, we will denote by $D(|s(t)|)$ the right derivative of $|s(t)|$. Then in the section $[t^*, T_{k-1}]$ we have

$$D(|s(t)|) \geq c|s(t)| - 5c\nu_k - 4c\nu_k/3; \quad |s(t^*)| = \nu_{k-1}.$$

Using Gronwall's lemma and assuming $t = T_{k-1}$, we obtain

$$(\nu_{k-1} - 19\nu_k/3)e^{c(T_{k-1} - t^*)} + 19\nu_k/3 \leq 0.$$

Hence it follows that

$$T_{k-1} - t^* \geq c^{-1}\ln(19/18).$$

We can similarly obtain an upper estimate of $|s(t)|$

$$D(|s(t)|) \leq c|s(t)| + 5c\nu_k + 4c\nu_k/3; \quad |s(T_{k-1})| = 0, \quad T_{k-1} < t < T^*.$$

From the last differential inequality we obtain

$$|s(t)| \leq 19\nu_k e^{c(t - T_k)}/3 - 19\nu_k/3, \quad T_{k-1} < t < T^*.$$

When $t = T^*$ we obtain

$$|s(T^*)| = \nu_{k-1}/\gamma \leq 19\nu_k e^{c(T^* - T_{k-1})}/3 - 19\nu_k/3.$$

Hence it follows that

$$T^* - T_{k-1} \geq c^{-1}\ln((19 + 1/\gamma)/19).$$

Further

$$T^* - t^* = T^* - T_{k-1} + T_{k-1} - t^* \geq$$
$$\geq c^{-1}\ln(19/18) + c^{-1}\ln((19 + 1/\gamma)/19) = c^{-1}\ln((19 + 1/\gamma)/18).$$

The inequalities $\gamma < 3$ and $ch_0 < \ln(15/14)$ imply $T^* - t^* > h_0$. So, according to the Property 2 $|s(\overline{T})| < \nu_{k-1}$, we have arrived at a contradiction. Consequently, $|s(t)| < \nu_{k-1}$ for all $t > T_{k-1}$.

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